Further simplification of the light deflection formula for solar system objects

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The transformation $\boldsymbol{n}$ to $\boldsymbol{k}$ in post-post-Newtonian order is simplified. All post-post-Newtonian terms of the order $\mathcal{O}\left(\frac{m^2}{d^2}\right)$ are neglected and we show that the total sum of these terms is smaller than $\frac{15}{4}\,\pi\,\frac{m^2}{d^2}$. This simpler transformation will improve the efficiency of Gaia data reduction.

I. INTRODUCTION

The approximative analytical solution of the problem of light deflection has been presented in [1–3]. One of the main result of these investigations is the transformation $\boldsymbol{n}$ to $\boldsymbol{k}$ for solar system objects in post-post-Newtonian approximation. A detailed analysis [3, 4] has shown that most of the terms in this transformation can be neglected at the microarcsecond level of accuracy, leading a simplified formula $\boldsymbol{n}$ to $\boldsymbol{k}$ for the data reduction. This simplified formula $\boldsymbol{n}$ to $\boldsymbol{k}$ has been given in Eqs. (92) and (93) in [1] and in Eqs. (52) and (53) in [3]. In this report we will show that this transformation can be further simplified. The report is organized as follows. In Section II we will present the transformation $\boldsymbol{n}$ to $\boldsymbol{k}$ in post-post-Newtonian order. The estimate of post-post-Newtonian terms and the new simplified transformation $\boldsymbol{n}$ to $\boldsymbol{k}$ in given in Section III. A new estimation will be given in Section IV. A summary is given in Section V. Detailed proofs of the estimates used are given in the appendices.

II. TRANSFORMATION n TO k IN POST-POST-NEWTONIAN ORDER

The transformation $\boldsymbol{n}$ to $\boldsymbol{k}$ in post-post-Newtonian order has been given in Eq. (87) in [1], Eq. (57) in [2], and in Eq. (45) in [3]. We will present this transformation in the following equivalent form:

$$
\begin{aligned}
\text{N}\quad & \boldsymbol{n} = \boldsymbol{k} \\
\text{pN}\quad & \quad - (1+\gamma)\, m\, \frac{\boldsymbol{k}\times(\boldsymbol{x}_0\times\boldsymbol{x}_1)}{x_1\,(x_1\,x_0+\boldsymbol{x}_1\cdot\boldsymbol{x}_0)} \\
\Delta\text{pN}\quad & \quad + (1+\gamma)^2\, m^2\, \frac{\boldsymbol{k}\times(\boldsymbol{x}_0\times\boldsymbol{x}_1)}{(x_1\,x_0+\boldsymbol{x}_1\cdot\boldsymbol{x}_0)^2}\,\frac{R}{x_1} \\
\text{scaling}\quad & \quad - \frac{1}{8}\,(1+\gamma)^2\,\frac{m^2}{x_1^2}\,\boldsymbol{k}\,\frac{\left((x_1-x_0)^2-R^2\right)^2}{|\boldsymbol{x}_1\times\boldsymbol{x}_0|^2} \\
\text{ppN}\quad & \quad + m^2\,\boldsymbol{k}\times(\boldsymbol{x}_0\times\boldsymbol{x}_1)\left[\frac{1}{2}\,(1+\gamma)^2\,\frac{R^2-(x_1-x_0)^2}{x_1^2\,|\boldsymbol{x}_1\times\boldsymbol{x}_0|^2}\right. \\
\text{ppN}\quad & \quad + \frac{1}{4}\,\alpha\,\epsilon\,\frac{1}{R}\left(\frac{1}{R\,x_0^2}-\frac{1}{R\,x_1^2}-2\,\frac{\boldsymbol{k}\cdot\boldsymbol{x}_1}{x_1^4}\right) \\
\text{ppN}\quad & \quad - \frac{1}{4}\,\left(\,8(1+\gamma-\alpha\gamma)(1+\gamma)-4\alpha\beta+3\,\alpha\,\epsilon\,\right)\,R\,\frac{\boldsymbol{k}\cdot\boldsymbol{x}_1}{x_1^2\,|\,\boldsymbol{x}_1\times\boldsymbol{x}_0\,|^2} \\
\text{ppN}\quad & \quad \left. + \frac{1}{8}\,\left(8(1+\gamma-\alpha\,\gamma)(1+\gamma)-4\,\alpha\,\beta+3\alpha\,\epsilon\right)\,\frac{x_1^2-x_0^2-R^2}{|\boldsymbol{x}_1\times\boldsymbol{x}_0|^3}\,\delta(\boldsymbol{x}_1,\boldsymbol{x}_0)\right] \\
\text{ppN}\quad & \quad + (1+\gamma)^2\,m^2\,\frac{\boldsymbol{k}\times(\boldsymbol{x}_0\times\boldsymbol{x}_1)}{(x_1\,x_0+\boldsymbol{x}_1\cdot\boldsymbol{x}_0)^2}\,\frac{x_1+x_0-R}{x_1} \\
& \quad + \mathcal{O}(c^{-6})\,.
\end{aligned}
\tag{1}
$$

Here we have classified the nature of the individual terms by labels N (Newtonian), pN (post-Newtonian), ppN (post-post-Newtonian) and ΔpN (terms that are formally of post-

post-Newtonian order, but may numerically become significantly larger than other post-post-Newtonian terms, see estimates in (6)).

III. SIMPLIFIED TRANSFORMATION $\boldsymbol{n}$ TO $\boldsymbol{k}$

The effect of all the "ppN" terms in (1) can be estimated as (cf. Eq. (91) in [1] or Eq. (50) in [3])

$$\left| \boldsymbol{\omega}'_{\rm ppN} \right| \le \frac{15}{4}\,\pi\,\frac{m^2}{d^2}\,. \tag{2}$$

The proof of (2) is given in Appendix A. These terms can attain $1\,\mu$as only for observations within about 3.3 angular radii from the Sun and can be neglected. Accordingly, we obtain a simplified formula for the transformation from $\boldsymbol{k}$ to $\boldsymbol{n}$ keeping only the post-Newtonian and "enhanced" post-post-Newtonian terms labelled as "pN" and "ΔpN" in (1):

$$\boldsymbol{n} = \boldsymbol{k} + \boldsymbol{d}\,P\,\left(1 + P\,x_1\right) + \mathcal{O}\left(\frac{m^2}{d^2}\right) + \mathcal{O}(m^3)\,, \tag{3}$$

$$P = -(1+\gamma)\,\frac{m}{d^2}\left(\frac{x_0 - x_1}{R} + \frac{\boldsymbol{k}\cdot\boldsymbol{x}_1}{x_1}\right). \tag{4}$$

The simplified transformation $\boldsymbol{n}$ to $\boldsymbol{k}$ given in Eq. (3) has now simpler structure than the former expression given in Eq. (92) in [1] or in Eq. (52) in [3]. Therefore, (3) is more efficient for the data reduction. Furthermore, the transformation in Eq. (3) has now similar structure as the simplified transformation $\boldsymbol{n}$ to $\boldsymbol{\sigma}$ given in Eq. (102) in [1] or in Eq. (62) in [3].

IV. A NEW ESTIMATION

The enhanced post-post-Newtonian term $\left|\boldsymbol{\omega}'_{\Delta pN}\right|$ in Eq. (1) is, for $\gamma = 1$, given by (cf. Eq. (89) in [1] or Eq. (48) in [3])

$$\left|\boldsymbol{\omega}'_{\Delta pN}\right| = 4\,m^2\,\frac{\left|\,\boldsymbol{k}\times(\boldsymbol{x}_0\times\boldsymbol{x}_1)\,\right|}{\left(x_1\,x_0 + \boldsymbol{x}_1\cdot\boldsymbol{x}_0\right)^2}\,\frac{R}{x_1}\,. \tag{5}$$

This term differs from the corresponding term $\left|\,\boldsymbol{\omega}_{\Delta pN}\,\right|$ defined in Eq. (89) in [1] or Eq. (48) in [3] only by a factor $\dfrac{R}{x_0 + x_1} \le 1$. Therefore, we conclude that the estimates given in Eqs. (89) and (90) of [1] or in Eqs. (48) and (49) of [3] are also valid for $\left|\boldsymbol{\omega}'_{\Delta pN}\right|$, that means:

$$\left|\boldsymbol{\omega}'_{\Delta pN}\right| \le 16\,\frac{m^2}{d^3}\,\frac{R^2\,x_1\,x_0^2}{\left(x_1 + x_0\right)^4} \le 16\,\frac{m^2}{d^3}\,\frac{R\,x_1\,x_0^2}{\left(x_1 + x_0\right)^3} \le 16\,\frac{m^2}{d^3}\,\frac{x_1\,x_0^2}{\left(x_1 + x_0\right)^2} \le 16\,\frac{m^2}{d^2}\,\frac{x_1}{d}\,, \tag{6}$$

where the first expression given in (6) represents a new estimation. Another estimation can be given, namely (cf. Eq. (90) in [1] or Eq. (49) in [3])

$$|\,\boldsymbol{\omega}'_{\Delta \rm pN}\,| \le \frac{64}{27}\,\frac{m^2}{d^2}\,\frac{R}{d}\,, \tag{7}$$

which cannot be related to the estimations in (6) and reflect different properties of $\left|\boldsymbol{\omega}'_{\Delta \rm pN}\right|$ as function of multiple variables.

V. SUMMARY

In Eq. (57) in [2] the complete transformation $\boldsymbol{n}$ to $\boldsymbol{k}$ in post-post-Newtonian order has been given. In [3] we have shown that most of the terms can be neglected because they are of the order $\mathcal{O}\left(\frac{m^2}{d^2}\right)$ and can attain $1\,\mu$as only for observations within about 3.3 angular radii from the Sun. These investigations have yielded a simplified transformation, given in Eqs. (92) and (93) in [1] or in Eqs. (52) and (53) in [3], and applicable for an efficient data reduction. In this report we have shown that Eq. (92) in [1] or Eq. (52) in [3] can further be simplified. The main result of this report is Eq. (3), where we give a new simplified transformation $\boldsymbol{n}$ to $\boldsymbol{k}$ which will improve the efficiency of Gaia data reduction. We have shown that the total sum of the neglected ppN-terms is smaller than $\frac{15}{4}\,\pi\,\frac{m^2}{d^2}$. Furthermore, estimations of the enhanced post-post-Newtonian term has been given in Eqs. (6) and (7).

[1] S.A. Klioner, S. Zschocke, *Numerical versus analytical accuracy of the formulas for light propagation*, Class. Quantum Grav. 27 (2001) 075015.

[2] Sergei A. Klioner, Sven Zschocke, GAIA-CA-TN-LO-SK-002-2, *Parametrized post-post-Newtonian analytical solution for light propagation*, available on arXiv:astro-ph/0902.4206.

[3] Sven Zschocke, Sergei A. Klioner, GAIA-CA-TN-LO-SZ-002-2, *Analytical solution for light propagation in Schwarzschild field having an accuracy of 1 micro-arcsecond*, available on arXiv:astro-ph/0904.3704.

[4] Sven Zschocke, Sergei A. Klioner, GAIA-CA-TN-LO-SZ-003-1, *Formal proof of some inequalities used in the analysis of the post-post-Newtonian light propagation theory*, available on arXiv:astro-ph/0907.4281.

Appendix A: Proof of inequality (2)

The sum of all ppN-terms in Eq. (1) can be written as follows (here $\alpha = \beta = \gamma = \epsilon = 1$):

$$\left| \boldsymbol{\omega}^{\,\prime}_{\rm ppN} \right| = \frac{1}{4}\, \frac{m^2}{d^2}\, f_{10}^{\,\prime}\,, \tag{A1}$$

where the function is defined by (cf. with f_{10} defined in Eq. (84) in [4])

$$f_{10}^{\,\prime} = \Bigg| \frac{z\,(16 z - z\,\cos\Phi - 15)\,\sin\Phi}{1+z^2 - 2z\,\cos\Phi} + \frac{z(1 - 3z^2 + 2z^3 \cos\Phi)\,\sin^3\Phi}{\left(1+z^2 - 2z\,\cos\Phi\right)^2} + \frac{15 z\,(\cos\Phi - z)\,\Phi}{1+z^2 - 2z\,\cos\Phi} + 16\,\frac{z\,\left(1-\cos\Phi\right)^2 \left(1 + z - \sqrt{1+z^2 - 2\,z\,\cos\Phi}\right)}{\left(1+z^2 - 2\,z\,\cos\Phi\right)\,\sin\Phi} \Bigg| . \tag{A2}$$

Here we have used the notation $\Phi = \delta\left(\boldsymbol{x}_0, \boldsymbol{x}_1\right)$ and $z = \frac{x_0}{x_1}$. By means of the inequalities (note that (A4) improves the inequality given in Eq. (C1) in [4])

$$f_2 = 16\,\frac{z\,\left(1-\cos\Phi\right)^2 \left(1 + z - \sqrt{1+z^2 - 2\,z\,\cos\Phi}\right)}{\left(1+z^2 - 2\,z\,\cos\Phi\right)\,\sin\Phi} \le\, 8\,\sin\Phi\,, \tag{A3}$$

$$f_3 = \frac{|\,z\,\left(1 - 3z^2 + 2z^3\cos\Phi\right)|\,\sin^3\Phi}{\left(1+z^2 - 2z\,\cos\Phi\right)^2} \le\, 3\,\sin\Phi\,, \tag{A4}$$

(proof of (A3) and (A4) are shown in Appendices B and C, respectively) we obtain

$$f_{10}^{\,\prime} \le \left| \frac{z\,(16 z - z\,\cos\Phi - 15)\,\sin\Phi}{1+z^2 - 2z\,\cos\Phi} + \frac{15 z\,(\cos\Phi - z)\,\Phi}{1+z^2 - 2z\,\cos\Phi} \right| + 11\,\sin\Phi\,. \tag{A5}$$

In [4] we have shown $z\,(16z - z\,\cos\Phi - 15)\,\sin\Phi + 15z\,(\cos\Phi - z)\,\Phi \le 0$. Accordingly, due to $\sin\Phi \ge 0$, we obtain

$$f_{10}^{\,\prime} \le \left| \frac{z\,(16 z - z\,\cos\Phi - 15)\,\sin\Phi}{1+z^2 - 2z\,\cos\Phi} + \frac{15 z\,(\cos\Phi - z)\,\Phi}{1+z^2 - 2z\,\cos\Phi} - 15\,\sin\Phi \right|\,, \tag{A6}$$

where, for convenience, we have replaced the term $11\,\sin\Phi$ by the larger term $15\,\sin\Phi$. Furthermore, in [4] we have shown that

$$\left| \frac{z\,(16 z - z\,\cos\Phi - 15)\,\sin\Phi}{1+z^2 - 2z\,\cos\Phi} + \frac{15 z\,(\cos\Phi - z)\,\Phi}{1+z^2 - 2z\,\cos\Phi} - 15\,\sin\Phi \right| \le 15\,\pi\,. \tag{A7}$$

Thus, we obtain

$$f_{10}^{\,\prime} \le 15\,\pi\,. \tag{A8}$$

The inequality (A8) in combination with (A1) shows the validity of inequality (2).

Appendix B: Proof of inequalities (A3)

In order to show (A3), we rewrite this inequality as follows:

$$\frac{z\,(1-\cos\Phi)}{1+z^2-2\,z\,\cos\Phi}\,\frac{1+z-\sqrt{1+z^2-2\,z\,\cos\Phi}}{1+\cos\Phi}\leq\frac{1}{2}\,. \tag{B1}$$

The inequality (B1) can be splitted into two factors satisfying the following inequalities:

$$\frac{z\,(1-\cos\Phi)}{1+z^2-2\,z\,\cos\Phi}\leq\frac{1}{2}\,, \tag{B2}$$

$$\frac{1+z-\sqrt{1+z^2-2\,z\,\cos\Phi}}{1+\cos\Phi}\leq 1\,. \tag{B3}$$

The inequality (B2) is obviously valid, because by multiplying (B2) with the denominator we obtain $-\,(1-z)^2\leq 0$. The inequality (B3) is also straightforward, because it can be rewritten as $z-\cos\Phi\leq\sqrt{1+z^2-2\,z\,\cos\Phi}$, which is obviously valid due to $z-\cos\Phi\leq|\,z-\cos\Phi\,|$. Thus we have shown the validity of inequality (B1) and (A3), respectively.

Appendix C: Proof of inequality (A4)

Using the notation $w = \cos\Phi$, the inequality (A4) can be written as follows:

$$f_3 = \frac{z\,|\,1-3\,z^2+2\,z^3\,w\,|\,(1-w^2)}{(1+z^2-2\,w\,z)^2} \leq 3\,. \tag{C1}$$

Using the inequality (proof see below)

$$\left|1-3\,z^2+2\,z^3\,w\right| \leq 1-3\,w\,z^2+2\,z^3\,, \tag{C2}$$

we obtain

$$f_3 \leq \frac{z\,(1-3\,w\,z^2+2\,z^3)\,(1-w^2)}{(1+z^2-2\,w\,z)^2} = h_1+h_2 \leq 3\,. \tag{C3}$$

In (C3) the relation $1-3\,w\,z^2+2\,z^3 = (1+z^2-2\,w\,z)+(-3\,w\,z^2+2\,z^3-z^2+2\,w\,z)$ has been used. The functions are defined by

$$h_1 = \frac{z\,(1-w^2)}{1+z^2-2\,w\,z} \leq \frac{2\,z\,(1-w)}{1+z^2-2\,w\,z} \leq 1\,, \tag{C4}$$

$$h_2 = \frac{z^2\,|-3\,w\,z+2\,z^2-z+2\,w\,|\,(1-w^2)}{(1+z^2-2\,w\,z)^2} \leq 2\,. \tag{C5}$$

The inequality (C4) has been shown in [4]. In order to show (C5), we factorize the function h_2 as follows:

$$h_2 = h_2^A\,h_2^B\,, \tag{C6}$$

$$h_2^A = \frac{z^2\,(1-w^2)}{1+z^2-2\,w\,z} \leq 1\,, \tag{C7}$$

$$h_2^B = \frac{|-3\,w\,z+2\,z^2-z+2\,w\,|}{1+z^2-2\,w\,z} \leq 2\,. \tag{C8}$$

Thus, by means of the inequalities (C2) and (C4) - (C8) we have shown the validity of inequality (C1) and (A4), respectively. We still have to proof of inequalities (C2), (C7) and (C8).

Let us consider (C2). First, we remark that $1-3\,w\,z^2+2\,z^3 \geq 0$ because of $1-3\,z^2+2\,z^3 \geq 0$. Then, squaring both sides of (C2) and subtracting from each other leads to

$$h_3 = 2\,z^3+2\,w\,z^3-3\,z^2-3\,w\,z^2+2 \geq 0\,. \tag{C9}$$

The boundaries of h_3 are

$$\lim_{w \to -1} h_3 = 2 \geq 0\,, \tag{C10}$$

$$\lim_{w \to +1} h_3 = 2\;(2\,z+1)\;(z-1)^2 \geq 0\,, \tag{C11}$$

$$\lim_{z \to 0} h_3 = 2 \geq 0\,, \tag{C12}$$

$$\lim_{z \to \infty} h_3 = 2\;(1+w)\;\lim_{z \to \infty} z^3 \geq 0\,. \tag{C13}$$

The extremal conditions $h_{3\,,\,w} = 0$ and $h_{3\,,\,z} = 0$ lead to

$$z^2\;(2\,z-3) = 0\,, \tag{C14}$$

$$z\;(1+w)\;(z-1) = 0\,. \tag{C15}$$

The common solutions of (C14) and (C15) are given by

$$P_1\,(w = -1\,,\; z = 0)\;, \tag{C16}$$

$$P_2\left(w = -1\,,\; z = \frac{3}{2}\right)\;. \tag{C17}$$

The numerical values of h_3 at these turning points are

$$h_3\,(P_1) = 2 \geq 0\,, \tag{C18}$$

$$h_3\,(P_2) = 2 \geq 0\,. \tag{C19}$$

Thus, we have shown (C9) and, therefore, the validity of inequality (C2).

Now we consider the inequality (C7). Multiplying both sides of this relation with the denominator leads to the inequality

$$h_4 = -z^2\,w^2 - 1 + 2\,w\,z \leq 0\,. \tag{C20}$$

The boundaries of h_4 are

$$\lim_{w \to -1} h_4 = -\,(1+z)^2 \leq 0\,, \tag{C21}$$

$$\lim_{w \to +1} h_4 = -\,(1-z)^2 \leq 0\,, \tag{C22}$$

$$\lim_{z \to 0} h_4 = -1 \leq 0\,, \tag{C23}$$

$$\lim_{z \to \infty} h_4 = -w^2\;\lim_{z \to \infty} z^2 \leq 0\,. \tag{C24}$$

The extremal conditions $h_{4\,,w} = 0$ and $h_{4\,,z} = 0$ lead to

$$z\,(1 - w\,z) = 0\,, \tag{C25}$$

$$w\,(1 - w\,z) = 0\,. \tag{C26}$$

The common solution of (C25) and (C26) is given by

$$P_3\,(w = 0\,,\; z = 0)\;, \tag{C27}$$

and the numerical value of h_4 at this turning point is

$$h_4\,(P_3) = -1 \leq 0\,. \tag{C28}$$

Thus, we have shown (C20) and, therefore, the inequality (C7).

Now we consider the inequality (C8). Squaring both sides of (C8) and subtracting from each other leads to the inequality

$$h_5 = 4\,z^2 - z - 7\,w\,z + 2 + 2\,w \geq 0\,. \tag{C29}$$

The boundaries of h_5 are

$$\lim_{w \to -1} h_5 = 2\,z\,\,(3 + 2\,z) \geq 0\,, \tag{C30}$$

$$\lim_{w \to +1} h_5 = 4\,\,(z - 1)^2 \geq 0\,, \tag{C31}$$

$$\lim_{z \to 0} h_5 = 2\,(1 + w) \geq 0\,, \tag{C32}$$

$$\lim_{z \to \infty} h_5 = 4\,\lim_{z \to \infty} z^2 \geq 0\,. \tag{C33}$$

The extremal conditions $h_{5\,,w} = 0$ and $h_{5\,z} = 0$ lead to

$$-\,7\,z + 2 = 0\,, \tag{C34}$$

$$8\,z - 1 - 7\,w = 0\,. \tag{C35}$$

The common solution of (C34) and (C35) is given by

$$P_4\left(w = \frac{9}{49}\,,\; z = \frac{2}{7}\right)\;, \tag{C36}$$

and the numerical value of h_5 at this turning point is

$$h_5\,(P_4) = \frac{100}{49} \geq 0\,. \tag{C37}$$

Thus, we have shown (C29) and, therefore, the inequality (C8).